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VISUAL THINKING AND THE SOCIO-HISTORICAL ASPECTS OF RICHARD DEDEKIND'S CONTRIBUTIONS TO THE FOUNDATIONS OF MATHEMATICS**

Abstract

This paper presents a new interpretation of Dedekind's philosophy of mathematics, based on an analysis of a selected part of his mathematical practice. The article consists of three parts. In the first part, I describe selected interpretations of Dedekind's philosophy of mathematics, such as fictionalism, creationism, or realism on the one hand, and the ontology of the intentional object or structuralism on the other. In the second part, I introduce the tools and methods that I use in the third part of the article, such as Giaquinto's proposed use of visual thinking in mathematical practice, as well as the socio-historical perspective of the development of mathematical knowledge. I also explain why a perspective of the philosophy of mathematical practice is useful here. In the last part of the article, I analyze how Dedekind introduced two number systems and present a new and original suggestion for the interpretation of his philosophy.

The main object of the research is Dedekind's mathematical texts and the mathematical concepts and objects they describe, but I also try to reinterpret the non-mathematical statements contained in these texts. Finally, I argue that in addition to certain "technical" arguments that can be made against fictionalism, realism, and creationism in Dedekind's case, the interpretation presented here can also be used as an argument from the perspective of cognitive psychology and the socio-historical perspective of mathematical practices.

Keywords: Richard Dedekind, real numbers, natural numbers, PMP, philosophy of mathematical practice, visual thinking, socio-historical practices

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INTRODUCTION

In the article, I propose solutions to some problems discussed by proponents of fictional and creationist interpretations on the one hand, and those that advocate for structuralism or the ontology of intentional objects on the other. These problems – although limited here to Richard Dedekind's case – are largely considered part of the traditional problems of the philosophy of mathematics. By expanding the traditional perspective through the lens of the philosophy of mathematical practice (PMP), a new interpretation of these problems is found.

There are many unsolved problems in the traditional approach to the philosophy of mathematics, such as the way mathematical objects exist and are created, including the discovery–creation dichotomy in mathematics. As an alternative (or complement) to the traditional philosophy of mathematics perspective, PMP is the main perspective in the article. This perspective addresses rather different issues and problems and solves them with different tools (Mancosu 2008). However, when mathematical practice is taken into account – for example, the mental or social dimension of a mathematical entity (agent-based view) – many of these problems cease to be problems or take on a different, weaker form.

For the present analyses, it is important to note that some authors interpret Dedekind's philosophical statements (for example, Jan Łukasiewicz, Roman Murawski, or Jerzy Dadaczyński), while others concentrate on the mathematical aspects of his texts (Piotr Błaszczyk, José Ferreirós, and Erich Reck). My aim is to take both aspects into account, but giving priority to the analysis of mathematical objects and concepts. It should also be noted that the interpretation presented in this work is more similar to those based on the mathematical parts of Dedekind's texts.

The distinction made above refers, in a way, to the difference noted by Michał Heller (2019) between philosophical doctrines about science that may accompany the attitudes of individual scientists regarding the theories they create (or discuss), and the philosophical doctrines that are directly involved in the scientific theories themselves. Using Heller's proposal to expand the range of issues that can be analyzed within the framework of "philosophy in mathematics," I will take into account the basic assumptions of PMP. Contemporary proposals by representatives of PMP argue that the philosophy of mathematics, when divorced from mathematical practice, may be too idealized and may not reflect the real aspects of the epistemology and ontology of mathematics (Carter 2019: 2).

The structure of the article is as follows: in the first part of the article, selected interpretive versions of Dedekind's philosophy are briefly presented and critically discussed. In the next section, the tools and perspectives for analyzing Dedekind's mathematical practice are presented, including some aspects related to visual thinking, as well as the socio-historical context of mathematical practices. Finally, I will present the most salient aspects of Dedekind's construction of a system of real (or rather irrational) and natural numbers; this will show that, without rejecting rigor in mathematics, Dedekind made effective use of the visual aspects associated with the perception of mathematical objects in combination with a pragmatic approach to problem solving.

The crux of my interpretation will be the claim that Dedekind had a visual category specification for a set of real numbers as well as a set of natural numbers when he introduced them, and that the construction of these specifications was made possible by the socio-historical context.

1. INTERPRETATIONS OF DEDEKIND'S PHILOSOPHY OF MATHEMATICS

In this part I will present some interpretations of Dedekind's philosophy of mathematics: fictionalism, creationism, a kind of realism, structuralism, and ontology of intentional objects. I will also describe the most important aspects that I am going to discuss.

In the Polish scientific community there is, in general, the constructivist interpretation of Dedekind (Dadaczyński 2000, Łukasiewicz 1915, Murawski 2001). This may have something to do with how Łukasiewicz (Łukasiewicz 1915: XXXIV) referred to Dedekind's own statement regarding the creation of numbers, but it did not result from Polish philosophers' preference for a particular trend in the philosophy of mathematics (Murawski 2004b).

In the context of traditional Polish philosophy of mathematics, constructivism is largely perceived as the subjective and arbitrary creation of mathematical content, where this content (or the concepts defining it) exists only in the mind of the creator. I do not dismiss the possibility of another interpretation of this concept, as it is outlined by Ferreirós (1999b) in the context of the approaches taken by the German school, which included Cantor, and the Göttingen group associated with Dedekind. In this paper, however, I would like to focus only on these two aspects of constructivism: fictionalism and creationism.

With the help of the PMP perspective, I will show that the assumption that mathematical content is introduced through mental, intentional construction *does not have to be associated with the subjectivism of knowledge* or with ontological fictionalism because objects introduced (or just analyzed) in individual mathematical practice *can also exist pragmatically in relation to socio-historical* practice.

1.1. CONCEPTUALISM, CREATIONISM, AND PLATONISM

The beginning of the creationist and even fictionalist interpretation of Dedekind in Poland was in 1914, when the Polish translation of Dedekind's *Stetigkeit und irrationale Zahlen* was published. I will therefore begin with a description of his position. However, since Łukasiewicz's comment on Dedekind consists of only one sentence, I will also try to give a brief overview of his broader approach to the philosophy of science.

Jan Łukasiewicz was a representative of the Lvov-Warsaw School; more precisely, he was a direct student of Kazimierz Twardowski (Brożek 2022). Representatives of various disciplines worked together in this scientific community – mathematicians, logicians, and philosophers (Murawski 2004b). Wacław Sierpiński, for example, was not only a mathematician but also had a keen interest in philosophical issues, as evidenced by his habilitation lecture on the subject, much like Zygmunt Janiszewski (Murawski 2004b). In the *Self-Study Guide* (Polish: *Poradnik dla samouków*, Michalski 1915), Łukasiewicz describes both Cantor's and Dedekind's methods for constructing real numbers and even provides a philosophical comparative analysis of these constructions.

As a logician, Łukasiewicz is best known for his development of multi-valued logic. However, in his works he also refers to questions in the field of the philosophy of science. For example, in (Łukasiewicz 1915) he described his approach to logic, as well as to empirical sciences and mathematics, which he treated very similarly to logic. In the context of Łukasiewicz's philosophy of mathematics, I consider two issues: the aspect of the source of knowledge of mathematics (i.e., the issue of the creation of mathematical content), and the aspect of the ontology of mathematical objects.

In the *Self-Study Guide* (Michalski 1915), a book devoted to mathematics, Łukasiewicz emphasized the creative role of the scientific subject in the creation and development of science, including logic and mathematics. Above all, science is "human creative work" (Łukasiewicz 1915: XV–XVI). In this context, he even compared science to art.

Interpreting Dedekind's statements about the free creation of numbers, Łukasiewicz emphasized the role of human mental activity, including creative activity, in the development of mathematics. Taking into account that Łukasiewicz believed that the only source of mathematical knowledge is the human mind (that knowledge has no – even indirect – connection with experience or reality, that it exists only in the mind), and that this knowledge about truth, or existence is not a mathematical object, it can be assumed that he considered the context of the ontology of mathematics irrelevant, at least.

Regarding the Platonism–nominalism dispute, Łukasiewicz is credited with a position closer to Platonism in the context of logic (Łukasiewicz 1937: 165; 1970: 249, Murawski 2014: 77). Although Łukasiewicz attributed real existence to entire sets of logical laws, he did not engage in the dispute over universals (Murawski 2004b: 332; 2014: 77). As Murawski emphasizes, Łukasiewicz pointed out that science is not just a set of true propositions, but a structure, each element of which must be related to other elements (Łukasiewicz 1916: 14–15, Murawski 2014: 69–70).

Moreover, as we know, Łukasiewicz's views evolved from a certain ontological realism towards relativism (however, as noted, realism was not concerned with objects themselves, but with systems composed according to logical laws). For example, he wrote that: "Logic is . . ., in my opinion, only an instrument which enables us to draw asserted conclusions from asserted premises" (Łukasiewicz 1952, Murawski 2004b).

At this point, it is worth noting more about Dedekind's sentence to which Łukasiewicz referred, and the context of Łukasiewicz's statement. Łukasiewicz wrote:

> Since the a priori sciences arose independently of experience, the creations of these sciences often have no connection with experience. In reality, there are no non-Euclidean geometric shapes or four-dimensional solids. But even a point, line, triangle, or cube of ordinary geometry does not exist in experience. A geometric point has no extent, and a line has only one dimension, length. Such creations probably do not exist in nature. All objects dealt with in geometry are ideal constructions of the mind. Similarly, there are no integers, rational, irrational, or complex numbers in the phenomenal world. According to Dedekind, numbers are "free creations of the human spirit." Numbers are the basis of all mathematical analysis. (Łukasiewicz 1915: XXXIV)

Considering only the traditional philosophy of mathematics, one might argue that Łukasiewicz, by quoting Dedekind's remarks on the creation of numbers, could be associated with the creationism or fictionalism approaches. However, from the perspective of PMP it is both natural and evident that the source of human knowledge lies in the mind or, more broadly, in individual and social scientific practices. In the context of ontology, as we will see, Łukasiewicz adopted a highly pragmatic approach, even though he attributed real existence to logical structures.

Murawski seems to go even further in his interpretation of Dedekind's philosophy. He implicitly links Dedekind to conceptualism and creationism, possibly as a reflection of his interpretation of Łukasiewicz's approach. Murawski writes:

> Intuitionism opposes Platonism and proclaims the ontological thesis of conceptualism, according to which mathematics is a function of the human intellect and the free life activity of reason. The objects studied by mathematics are concepts that exist in the mind. (Murawski 2003: 77)

If we paraphrase Murawski's statement about intuitionism and substitute intuitionism with the notion of constructivism attributed to Dedekind (intuitionism being a particularly radical form of constructivism in the sense described above), we get a definition of this perspective which stands in stark opposition to the Platonism associated with Cantor in the context of ontology. Murawski, moreover, directly attributes to Dedekind the perspectives of conceptualism (a form of fictionalism) as well as creationism. He writes:

> Proponents of conceptualism can be found in different historical periods. Among them, for example, was . . . the great German mathematician Richard Dedekind. . . . the creator of the formal theory of real numbers proclaimed, for example, that numbers are a free product of the human mind (when writing about the creation of irrational numbers, he used the German word *erschaffen*, which means "to create" and which appears in the Book of Genesis when describing the creation of the world by Yahweh). (Murawski 2004a: 256–257)

Murawski refers to the concept of *erschaffen*, which "Dedekind used when writing about the creation of irrational numbers" (Murawski 2004a: 257). The second of Dedekind's important statements about the free creation of numbers, however, was introduced in the context of the arithmetic of natural numbers in *Was sind und was sollen die Zahlen?* (Dedekind 1888).

In his analysis of Hilbert's arithmetic, Dadaczyński examined the ontology of mathematical objects in Dedekind's work, offering a slightly different interpretation of the sentence from Dedekind's 1888 paper. He claims that:

> the remark about the "free creations of the human spirit" determines . . . the existence of abstract arithmetic objects independent of "input" objects. (Dadaczyński 2012: 106)

But he also explains that:

> The objects of arithmetic are abstract objects because they come into being by "freeing" ("abstracting") them from "all other content" (all properties), except properties that are conditions imposed on the successor function. (Dadaczyński 2012: 106)

Finally, Dadaczyński writes that Dedekind's objects of arithmetic can be both abstract and non-abstract. This, however, is not entirely convincing when we consider the strict context of Dedekind's mathematical practice.

This interpretation of Dedekind's ontology seems too strong, since Dedekind did not consider in his texts the ontological properties of constructed number models as well as their individual elements. His construction does not determine the detailed ontology of mathematical objects or the independence of their existence. It also seems that Dedekind regarded natural numbers, ordinal numbers, and any other entities that satisfy the conditions of his constructed set theory-based model as distinct and separate objects.

Błaszczyk's interpretation of Dedekind's ontology is probably closest to my understanding because it is based on an analysis of Dedekind's 1872 mathematical text and the objects described in it. However, since Błaszczyk's interpretation focuses solely on the question of ontology, I will also explore the position of Ferreirós and Reck (2020) regarding Dedekind's methodological-epistemological structuralism, which, as the name suggests, addresses epistemology and methodology within Dedekind's mathematical practice. It is worth noting that the perspective of my analysis is primarily focused on individual mental mathematical practice.

1.2. INTENDED MODEL AND STRUCTURALISM

The perspective of PMP adopted in this article is drawn from Jessica Carter's (2019) work. In some respects, Błaszczyk's (2005, 2007) proposal, which was outlined in his habilitation thesis and summarized in an article, already presents certain aspects of the PMP framework in relation to Dedekind's construction of the set of real numbers.

Błaszczyk (2005, 2007: 171–176) argues that the category of mathematical objects proposed by Dedekind (the set of real numbers with the property of continuity) is, in a derivative sense, an intentional object in the Ingardenian framework.

He also categorizes Cantor's construction as belonging to the realm of intentional objects. However, Błaszczyk does not delve into the question of ontology in the traditional philosophical sense. He does not attempt to determine whether the mathematical objects introduced by Dedekind exist as Platonic entities or as invented literary fictions. Instead, his proposal can be described as pragmatic.

Since Błaszczyk compares the ontology of mathematical objects to that of fictional entities described in literary texts (e.g., novels), his approach can be seen as aligning closely with fictionalism. However, as Burgess's argument against nominalism and fictionalism highlights, there is a difference between mathematics and art: fictional characters appear in a work, whereas mathematical entities exist in different mathematical theories (cf. Burgess 2004, Horsten 2021). In this context, it is worth mentioning that for Łukasiewicz the common element of these two perspectives was creativity, while the distinguishing elements were aesthetic and intellectual needs.

Upon closer and more detailed examination, Błaszczyk's proposal is not connected to fictionalism in the sense of the naive understanding of literal fiction. Moreover, Błaszczyk argues for the intersubjective existence of mathematical objects, emphasizing that intersubjectivity is a feature of derivatively intentional objects, whereas only primarily intentional objects lack intersubjectivity. Within the context of PMP, this highlights the intrinsic connection between mathematics and mathematical practice, i.e., that mathematical objects are derived from this practice (Błaszczyk 2005).

As previously mentioned, Błaszczyk focuses on ontology rather than epistemology in his analyses of Dedekind's work. Therefore, to briefly outline an epistemological perspective that aligns most closely with the interpretation presented in this article, I will discuss Ferreirós and Reck (2020).

In the context of the method used to introduce mathematical content, the epistemic-methodological structuralism attributed to Dedekind deserves attention (Corry 2004, Edwards 1983, Ferreirós 1999a, Ferreirós and Reck 2020). As Ferreirós and Reck explain, the structuralism in question:

> is a style of work that takes results in a given branch of mathematics to emerge from the nature of relevant structures (exemplified therein), and, often typically, from certain interrelations between structures of different kinds. (Ferreirós and Reck 2020: 60)

In Dedekind's work, it is noted that the methodological-epistemological structuralism attributed to him was shaped by his experience with the practice of algebra (particularly Galois's theory), at that time a highly abstract field of mathematics also referred to as general arithmetic with symbolic calculus (Ferreirós and Reck 2020: 70).

With regard to the construction of his numerical models (N and $\mathbb{R}$), it is noted that Dedekind aimed to ground these models on logical laws, leading some to regard him as a logician (Ferreirós 1999c: 255). By emphasizing the classical nature of Dedekind's theory and the possibility of interpreting his statements about the creation of numbers in the mind in light of Kant's philosophy, it is also argued that he refutes subjective constructivism (Ferreirós and Reck 2020: 77).

I propose a new interpretation of Dedekind's philosophy that is grounded in analysis of mathematical structures and insights from cognitive psychology. This interpretation frames structuralism as the foundation for the methodology and epistemology of Dedekind's mathematical practice, while also enabling a more universal understanding of abstraction and construction in his texts. The basis of this interpretation lies in the assertion that Dedekind was guided by a mental specification of the visual (or partially visual) category of the structures he constructed. I will show that Dedekind constructed numerical models (N and $\mathbb{R}$) using the tools of theory, much like phenomena of the external world – whether natural or artificial – are modeled using mathematical tools (Bender 1942). This position will be supported by arguments that highlight Dedekind's awareness of his goals, his consistency, and his focus on the most essential and necessary aspects of a given structure. These arguments also emphasize how these aspects are integrated into a whole, and, ultimately, the categoricality of models (whether potential or achieved under specific conditions), viewed as a measure of their effectiveness and, therefore, their correctness.

1.3. NEW INTERPRETATION

The following "technical" arguments can be made against fictionalism, creationism, or realism:

(1) Proposals of fictionalism or creationism do not fit the interpretation of the mathematical part of Dedekind's work because we can interpret the texts from 1872 and 1988 without interpreting Dedekind's views. In fact, mathematicians have taken two ideas from the work of 1872, namely the cut of a linear ordered set and the principle of continuity (in the form of the cut or the supremum principle), without even entering

into interpretation of Dedekind's claims about the creation of irrational numbers.

(2) These theories do not fit the problem posed by Dedekind because they deal with real numbers as a whole; instead, Dedekind writes about creating irrational numbers. Real numbers are made up of rational and irrational numbers. This is far from the conclusion that real numbers are created; in the 1872 work, rational numbers are neither constructed nor described axiomatically.[1]

(3) The ontological realist interpretation of Dedekind's philosophy appears to be the least plausible. Dedekind does not make any statement that could serve as a basis for such an interpretation.

I propose an interpretation of Dedekind's philosophy that is grounded in an analysis of his mathematical practice. While this proposal aligns with the positions of Ferreirós and Reck, as well as Błaszczyk, it uses different and additional tools to develop its framework. Building on these analyses of Dedekind's mathematical practice, I propose a different understanding of Dedekind's statements about the creation of numbers – one that differs from the interpretations underpinning creationist and fictionalist views (e.g., Murawski) as well as realist perspectives (e.g., Dadaczyński).

According to the fundamental assumptions of PMP, mathematical practice encompasses both the theoretical aspects of mathematics itself and the activities of mathematicians (Carter 2019). Consequently, in addition to the analyses of Dedekind's texts from 1872 and 1888, I will propose a reconstruction of the cognitive process that might have been involved in the construction of Dedekind's mathematical structures.

The basis of my argument will be an analysis of Dedekind's texts and the possible mental processes involved in their construction. In particular, I will argue that it is highly probable that Dedekind was guided by a visual category specification of the number structures he constructed, and that his primary goal was to build a mathematical structure to address specific mathematical problems. Furthermore, I will demonstrate that the elements of these structures were linked to certain aspects relevant within the socio-historical context, yet Dedekind imbued them with a meaning consistent with the purpose of the constructed structure.

[1] I would like to thank one of the anonymous reviewers for this insightful comment.

Thus, the main aspects of the philosophy of mathematics that can be reconstructed through an analysis of Dedekind's mathematical practice (i.e., through analysis of textual fragments, his method of introducing numerical models, and certain properties of these models) are a form of two-dimensional pragmatic ontological realism and epistemic-methodological structuralism (Ferreirós and Reck 2020), here interpreted through the lens of set-theoretical modeling. I will relate such a structuralism to individual mathematical practice and to the socio-historical perspective (Carter 2004, 2022, Chateaubriand 2012, Hartimo and Rytilä 2023, Król 2003).

2. PMP: VISUAL THINKING AND THE SOCIO-HISTORICAL CONTEXT

In this chapter, I will briefly introduce the tools I use to analyze Dedekind's mathematical practice. The first is visual thinking, the second is the socio-historical context. Both tools will be used to construct an argument that challenges the fictionalist, creationist, and realist interpretations of Dedekind's philosophy presented at the beginning of this article.

2.1. VISUAL THINKING

Visual thinking, or visual aspects more generally, can be treated as an argument in favor of a version of pragmatic realism for mathematical objects. Thanks to these abilities, we are able to recognize a symbolically described structure or a single element as a separate object. Even if they do not exist independently of mathematicians (they arise as a result of their practices and are transmitted through texts and communication), we can speak of a sense of their existence in the context of both individual practice and socio-historical practice (through texts and communication).

I will argue that Dedekind's mathematical practice – at least in terms of the foundations of mathematics – involved set-theoretic modeling. He built a discrete-continuum model of the real numbers, a model of the natural numbers, and one using set-theoretic concepts and relations. However, I can agree with Carter (2008) that the ontology of mathematics cannot be reduced to structures themselves.

Mathematical modeling is typically associated with non-mathematical problems and objects (Bender 1942, Mityushev, Nawalaniec, and Rylko 2018, Rylko and Tytko 2022). When writing about a numerical model in the context of the modeling method, we do not intend to compare mathematics with other sciences on an equal basis, but only to exploit the fact that the concept of a model can be understood in the same way in the contexts of mathematics and of the empirical sciences (Suppes 1969: 12). A scientific model is a representation of selected important aspects of a fragment of a particular reality. The measure of the correctness of the construction of a given model is its effectiveness in application to the problems for which it was constructed (Bender 1942: 1).

There are four steps to building a mathematical model:

(1) Formulating the problem,

(2) Constructing the model (deciding what should and should not be considered),

(3) Checking the usability of the model,

(4) Testing the model (Bender 1942: 6–7).

I will now try to explain how the process of constructing a mathematical model can be described from the perspective of the mental level. I am using here the proposition of visual thinking in mathematics, which has also been analyzed by Marcus Giaquinto.

Giaquinto claims, for example, that "in some cases, we can know structures in more intimate ways, by means of our visual capacities" (Giaquinto 2008b: 44). He also describes two important mental representations:

(1) **visual image** – "a transient item of visual experience of a specific phenomenological type" (Giaquinto 2008a: 54),

(2) **visual category specification** – "a stored representation, consisting of an ensemble of feature descriptions" (Giaquinto 2008a: 54).

The visual category specification can produce, by shifting attention, the simple images that give us the structure described from one perspective. Of course, these images are not this specification, but the specification can guarantee that we can get any possible image that describes this structure.

If the structure to be analyzed is a structured set, then the most important thing is to select the most important elements and relations to be considered, where "a structured set is a set considered under one or more relations, functions, distinguished elements (constants) or some combination of these" (Giaquinto 2008a: 43–44). Since various models are used in concrete problems, it is also important to know the problem and how it is related to the purpose of the created structure, as well as how the purpose is related to the selected elements and relations. Of course, the ability to build a model can be affected by the visual skills of certain scientists. As Bender, the author of *An Introduction to Mathematical Modeling* writes:

> Giving rules for doing it [model building] is like listing rules for being an artist; at best this provides a framework around which to build skills and develop imagination. It may be impossible to teach imagination. (Bender 1942: 6)

I argue that Dedekind knew what aspects to consider in order to build a model to solve a particular problem in the context of the two number systems – $\mathbb{R}$ and N. In other words, I argue that Dedekind created (in a more or less conscious way) the visual category specification of these structures. This was related to a particular mathematical problem, as well as to the visual aspects associated with that problem and its solution. Fictionalism rejects the existence of mathematical objects, whereas Dedekind took into account some kind of pragmatical (visual) ontology of analyzed and introduced objects.

2.2. SOCIO-HISTORICAL CONTEXT

The second aspect of my argument is related to the claim that we can find elements in the visual category specification that are related to socio-historical aspects, such as a straight line composed of points, or the successor function. Of course, Dedekind defined some elements differently than they had been defined before (e.g., continuity), but he formalized them according to certain problems which were at the same time the purpose of the constructed structure. This argument can be used against both creationism and fictionalism.

As I have already mentioned, in this paper I am considering the term "constructivism," which is related to fictionalism and creationism. Instead of "constructivism" I would like to propose the term"constructionism" as it better fits Dedekind's mathematical practice. Taking into account Dedekind's originality and creativity in solving mathematical problems, his approach to mathematical practice should be considered a kind of constructionism, which is a broader (and weaker) concept than constructivism (Zwierżdżyński 2012) that – in addition to focusing on the mental constructions of an individual subject – also takes into account the historical and social context of mathematical knowledge (Charmaz 2006). We can reconcile this proposal with the pragmatic realism given to Dedekind's mathematical objects, and thanks to PMP's pragmatic approach we can go beyond the traditional Platonism–creationism dichotomy (Carter 2004, Chateaubriand 2012). This approach means that mathematical objects are created through mathematical practices, both in individual and socio-historical practices, and they only exist in this context.

Julian C. Cole, for example, claims that:

> The freedom and authority that mathematicians feel they enjoy to creatively postulate mathematical entities do not accord well with realist or Platonist interpretations of mathematical theories. Yet, the intellectual ease with which mathematicians ontologically commit themselves to mathematical entities does not accord well with fictionalist or modal nominalist interpretations of mathematical theories. (Cole 2009: 593, Rytilä 2021)

Cole proposes looking at the process of introducing and existing mathematical objects through the perspective of a working mathematician. He explains that "postulation is guided by the problem to be solved and a need for coherence" (Cole 2009: 591, Rytilä 2021). In the context of Dedekind's individual mathematical practice, which is determined by the modeling method, the above seems to be confirmed.

The creationism and fictionalism that we have discussed in the context of Dedekind's words, namely "numbers are a free creation of the human spirit," may be inadequate if we consider Dedekind's entire mathematical practice. We can look for and find elements that point to the undeniable role of the subject in the creation of mathematics, such as his construction of a category specification for structures that are solutions to certain mathematical problems. However, we should note that this is not a simple and direct "production" of numbers by the mind, as understood by classical constructivism in mathematics (Bridges, Palmgren, and Ishihara 2022), but rather the construction of structures based on existing objects and methods. Dedekind treated set theory as a tool for explaining and describing selected mathematical problems. This "creation of numbers" should not be seen in the context of the biblical term *erschaffen*, but in the context of creating a model that includes socio-historical elements.

3. DEDEKIND'S MATHEMATICAL CONSTRUCTION OF STRUCTURES

In this part of the paper, I would like to show, first, that Dedekind constructed category specifications of real and natural numbers in order to solve specific mathematical problems, that he relied on visual intuition in this aspect, and that he relied on existing (previously introduced) objects and accepted methods. Visual thinking, as I have mentioned, will be an argument for pragmatic realism, and the socio-historical context will be an argument for epistemological constructionism.

3.1. REAL NUMBERS

Dedekind introduced the properties of the set of real numbers, which were composed of the properties of rational numbers and the continuity property. What is most important for analyses of visual thinking and the creation of visual category specifications is that Dedekind described the same aspects of the structure being created both at the beginning and at the end of his 1872 work. What distinguishes these descriptions is their degree of formalization: at the beginning, he described the property of continuity intuitively, while at the end he gave it a formal definition. Importantly, in both cases he was also referring to properties describing the ordered set of rational numbers (their linear order). At the beginning he wrote this:

> It is often said that differential calculus deals with continuous quantities, but nowhere is any explanation of this continuity given; moreover, even in the strictest lectures of this calculus, the proofs are not based on continuity but refer either to certain more or less geometrical or geometrically suggested ideas, or to certain theorems which have never been proved in a purely arithmetical way. (Dedekind 1872: 316)

And:

> I find the essence of continuity . . . in the following principle: "If all points in a line are decomposed into two classes such that each point in the first class is to the left of any point in the second class, then there exists one and only one point, which produces this division of all points in two classes, this cutting of the line in two parts." (Ferreirós 1999b: 133, Dedekind 1872: 322)

It should also be mentioned that Dedekind took his idea for defining continuity directly from the problem of irrational numbers, or rather from the lack of such numbers. In other words, in his specification of the visual category, Dedekind referred to a straight line and identified numbers with points on it. With his property of continuity – as far as visual aspects are concerned – he ensured that the gaps in the set of irrational numbers would be "filled in." He wrote this himself:

> Whenever one finds a cut (A_1, A_2) which is not produced by any rational number, we create a new irrational number α, which we regard as completely defined by this cut (A_1, A_2); we shall say that the number α corresponds to this cut, or that it produces this cut. (Ferreirós 1999b: 134, Dedekind 1872: 325)

The essential idea of Dedekind's considerations about any number that is not produced by a cut in the set of rational numbers but is produced in the set of real numbers can be visualized by the following example with the help of a diagram with a symbolic description:

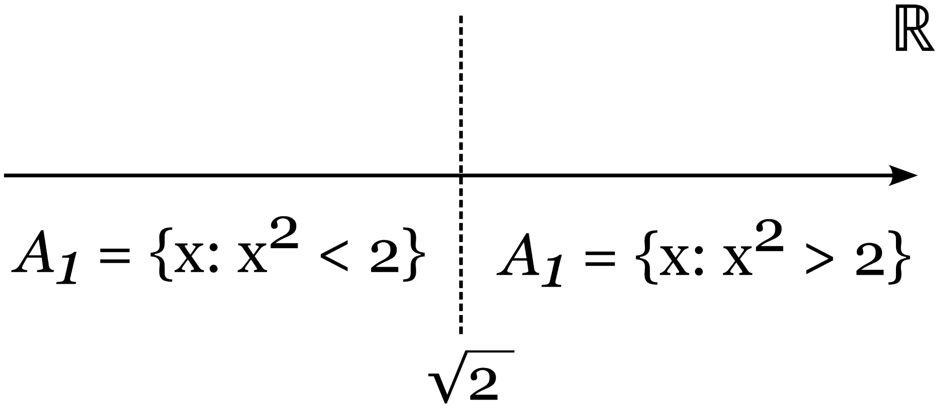

Figure 1. The Dedekind cut on the set of real numbers, equal to the root of two. This cut on the set of rational numbers is a gap.

The ontological aspects here can be understood in a pragmatic way. The pragmatic existence of abstract objects is possible because they are the subject of mathematical discourse – in both individual and socio-historical practice (Carter 2022, Chateaubriand 2012). I would like to note that the mental mathematical objects related to the numerical systems under discussion existed in relation to both Dedekind's mathematical practice and the socio-historical perspective.

Dedekind was aware of the relationship between analysis and geometry on a mental level. Hence, although he wanted to make the construction of real numbers independent of geometric intuition, in defining the property of continuity he referred to an intuitive – as it was perceived, for example, by Bolzano (Błaszczyk and Fila 2024) – straight line composed of points. He first showed this idea on a straight line composed of points and then defined this property for the set of real numbers. In the same way, he showed the incompleteness of the set of rational numbers. This is how he imagined this property in the above context:

> If all points of a straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions. (Dedekind 1872: 322; 1963: 11)

This idea (some visualization) was the basis for his definition of a set cut, according to which he classified a given set as having or not having the property of continuity:

> If the system $\mathbb{R}$ of all real numbers breaks up into two classes, A_1, A_2, such that every number α_1 of class A_1 is less than every number α_2 of class A_2, then there exists one and only one number α by which this separation is produced. (Dedekind 1872: 329; 1963: 19)

One example of Dedekind's reliance on a visual category specification in formalizing the property of continuity is the clear availability of a visual representation of the cut concept that he defined (Dedekind's cut). One can assume that Dedekind creatively combined a certain visual intuition about the concept of a cut with rational expectations about it. The concept of a cut is the basis of the continuity property he describes. Cuts in an ordered field of the real numbers can be represented as follows (Błaszczyk and Fila 2020: 46):

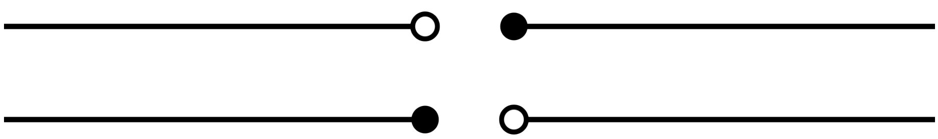

Figure 2. The general visual idea of the principle of Dedekind's cut on the set of real numbers (see Błaszczyk and Fila 2020: 46).

In the context of our postulate about the specification of a visual category for the structure of the set of real numbers, it is significant that Dedekind was the first to present the continuity property in terms of the theory of linearly ordered sets. Finally, Dedekind presented the same three properties of the set of rational numbers as properties of the set of real numbers and a formalized continuity property:

> I. If $\alpha > \beta$, and $\beta > \gamma$, then is also $\alpha > \gamma$. We shall say that the number β lies between α and γ.
>
> II. If α, γ are any two different numbers, then there exist infinitely many different numbers β between α, γ.
>
> III. If α is any definite number, then all numbers of the system $\mathbb{R}$ fall into two classes, A_1 and A_2, each of which contains infinitely many individuals; the first class, A_1, comprises all the numbers α_1 that are less than α; the second class, A_2, comprises all the numbers α_2 that are greater than α; the number α itself may be assigned at pleasure to the first class or to the second, and it is respectively the greatest of the first or the least of the second class. In each case the separation of the system $\mathbb{R}$ into the two classes A_1, A_2 is such that every number of the first class, A_1, is smaller than every number of the second class, A_2, and we say that this separation is produced by the number α.
>
> For brevity and in order not to weary the reader, I suppress the proofs of these theorems which follow immediately from the definitions of the previous section. Beside these properties, however, the domain $\mathbb{R}$ also possesses continuity, i.e., the following theorem is true:
>
> IV. If the system $\mathbb{R}$ of all real numbers breaks up into two classes A_1, A_2 such that every number α_1 of class A_1 is less than every number α_2 of class A_2, then there exists one and only one number α by which this cut is produced. (Dedekind 1872: 328–329; 1963: 19–20)

The first two points describe the transitivity and density of the order of numbers. Point III describes the Dedekind cut itself, while point IV describes the continuity property. It can also be observed that from III we can obtain the law of trichotomy.

The aspects that Dedekind took into account in the mental category specification of this model were a system that satisfies conditions analogous to the operations and order specified in the system of rational numbers, and, additionally, the property of continuity. The confirmation of this model's usefulness can be seen as the fulfillment of Dedekind's intention (building a discrete continuum), and its correctness – the fact that his continuity property is the basis of the infinitesimal analysis. However, from the contemporary point of view, we can also obtain the categoricity of the ordered field with Dedekind's continuity property (Błaszczyk 2005, 2007: 15; 2012: 23).

Again, Dedekind does not create numbers in a purely constructivist sense. Błaszczyk writes that:

> An irrational number is "created" because such aspects are "created," defined, due to which cuts of the set ($\mathbb{R}$, <) are numbers. (Błaszczyk 2007: 42)

I argue that the concept of "creation" can be interpreted in a broader sense, namely as mentally defining ("creating") aspects of the real number model, by which the existence of irrational numbers can be constituted. It is about the possibility of formally introducing irrational numbers into mathematics, but it is also about a certain redefinition within this model of the field of rational numbers $\mathbb{Q}$, as a subfield of fractions of the field $\mathbb{R}$. In this context, the model itself, real numbers, rational numbers, can be different objects.

In the above context, although constructing these models required Dedekind to engage in mental construction (Carter 2004), neither natural numbers nor real numbers are simply created (in the sense of the word *erschaffen*) by the human mind but are constituted in complex structural models. Epistemic-methodological structuralism in this context means that Dedekind was aware that he was defining structures (models) on the basis of already created knowledge and that this shaped his methodology (Reck 2020, Tait 1996), which we here call a method of set-theoretic modeling.

Finally, it can be said that Dedekind referred to elements of a certain area of pre-existing mathematical knowledge (the problem of the gaps in the set of rational numbers), but he created an original definition of the continuity property based on geometric visual aspects, then he formalized it in the set of real numbers. Moreover, it can be assumed that Dedekind used the analysis of the visual representation of – more or less formalized – mathematical objects and their properties as a basis for the introduced model.[2]

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[2]At this point we can refer to Giaquinto's proposal, which distinguishes between direct mental visual-intuitive experience of a mathematical object and a fully defined, complete specification of the category of visual representations, which can be the source of many different images but sometimes needs descriptive clarification. The latter seems especially necessary for more complex structures. Although Giaquinto states that a visual experience of the essence of the set R is not possible, one can consider whether it is possible to understand it indirectly through the visualisation of a Dedekind cut (Giaquinto 2008a, 2008b).

3.2. NATURAL NUMBERS

In this part, I will also discuss aspects that help to show that in the case of the model that Dedekind constructed for the formal introduction of natural numbers, he was also guided by a visual category specification, and that this specification referred to elements of understanding these numbers in a socio-historical context.

First of all, it should be noted that Dedekind, in the context of the set of natural numbers, used his set theory, laid out in the work of 1888. At the beginning, Dedekind introduces the basic elements of set theory (such as things, i.e., objects of thought, sets, i.e., systems created by gathering together objects of thought, as well as mappings, which are rules but refer to the activity of assigning values to the elements of a set), then he consistently and precisely introduces subsequent more complex set-theoretical objects (such as *Gemeinheit* or *Kette*), and finally he arrives at the description and presentation of a simply infinite set, which he presents as a model of the set of natural numbers (Dedekind 1888: 344, 356, 359).

Why did Dedekind construct a set-theoretic model for the arithmetic of natural numbers? Because, as he wrote early on in his work of 1888:

> In science nothing capable of proof ought to be accepted without proof. . . . If we scrutinize closely what is done in counting an aggregate or number of things, we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible. . . . Upon this unique and therefore absolutely indispensable foundation, as I have already affirmed in an announcement of this paper, the whole science of numbers must, in my judgment, be established. (Dedekind 1888: 335; 1963: 31)

What is also important here is that the basic element of the simple infinity set was the chain. Where:

> **Definition.** K is called a chain [*Kette*], when K' ⊆K. We remark expressly that this name does not in itself belong to the part K of the system S, but is given only with respect to the particular mapping φ; with reference to another transformation of the system S, in itself K can very well not be a chain. (Dedekind 1888: 353; 1963: 56)

Dedekind explains that the simply infinite set he constructs in this way always allows the same relations and laws to be satisfied in this system, regardless of the names of the elements of a given system, as long as the system satisfies the conditions of a simply infinite set. Of course, Dedekind also mentions that by abstracting numbers from each of their properties, we can admit that these numbers are created; he calls them "free creation of the human mind":

> A set N is said to be simply infinite when there exists a one-to-one mapping $\varphi: N \to N$ such that N is the chain . . . of an element not contained in the image $\varphi(N)$. We will call this element the base element and denote it 1. (Dedekind 1888: 359, Joyce 2005: 15)

But this sentence is more about the possibility of creating a model for natural numbers, and of creating numbers in the sense of constituting them in this model as abstract objects of the model. It is not about his belief in the creative power of the human mind to literally create (*erschaffen*) numbers. As for his statement that numbers are "creations of the mind," this can be interpreted – in the context of the above – as the mental ability to construct and work with abstract structures. In this sense, Dedekind's constructivism – or rather constructionism – is a direct consequence of his structuralist approach to his mathematical practice. In this sense, the mind can produce a new meaning and understanding of numbers. Joyce writes that "a simply infinite set is a model of the natural numbers" (Joyce 2005: 15).

This is how Dedekind expressed his intention and aim to define – using set theory and combining in a structure – the first element, the successor function, and the system, which according to him characterizes the set of natural numbers N, and the simply infinite set. The confirmation of the correctness of his model is – under certain conditions – its categoricity (Joyce 2005: 15), and the usefulness is related to the achievement of the intended goal (i.e., to formulate the arithmetic of N in a more transparent and precise way).

In the case of natural numbers, as in the case of real numbers, Dedekind was referring to the elements and problems of a particular area of pre-existing mathematical practice. Here it was the problem of the lack of a formal theory of the arithmetic of natural numbers that he created. But – despite the fact that he mentioned his intention to avoid geometrical intuitions – he used a reference to intuitive objects of thought. Moreover, the basic element of his system of natural numbers – the chain – can be represented in a visual way:

I argue that the proposed constructionism in the context of the Dedekind's mathematical practice does not contradict the creative activity of human beings, as emphasized, for example, by Łukasiewicz. In the PMP approach, pragmatic realism means that mathematical objects exist in a way that is derivative of

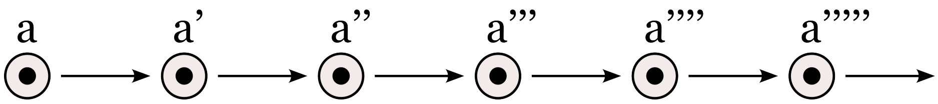

Figure 3. The chain defined by Dedekind in 1888. This chain (*Kette*) may present a set of natural numbers (see Joyce 2005: 15).

mathematical practice (Carter 2004, 2022). At the same time, this practice is immersed in the space of previously constructed objects, relations, and structures.

CONCLUSIONS

In summary, this article presents an interpretation of Dedekind’s philosophy in terms of visual thinking and the socio-historical perspective. This interpretation can be seen as an additional argument against fictionalism and creationism, with which Dedekind is associated in the Polish philosophy of mathematics community. It is also an argument in favor of the more obvious anti-Platonist interpretation. This argument is related to structuralist interpretations and those related to the way of existence of intentional objects. The perspective adopted here is one that goes beyond the traditional philosophy of mathematics and uses more research tools thanks to the PMP perspective.

I have proposed attributing a kind of epistemic constructionism rather than traditional constructivism to Dedekind’s mathematical practice. Dedekind referred to certain problems, objects and their properties that already existed in the realm of objective mathematical practice. He did not create mathematical structures in an unlimited and completely arbitrary way. Therefore, the creationism attributed to Dedekind’s philosophy is not entirely accurate. He solved specific problems using available methods and pre-existing mathematical objects. In such a context, constructivism – in addition to the activities of a given mathematical entity – also emphasizes the influence of the historical and social perspective on that entity, which traditional constructivism does not do (Charmaz 2006, Hartimo and Rytilä 2023).

Secondly, in response to the fictionalistic (conceptualistic) aspect attributed to Dedekind, I have proposed a certain kind of pragmatic realism, referring to the pragmatic proposals of PMP. Dedekind introduced new objects into mathematical theory (i.e., two models of sets of numbers) that were based on objects already existing in the historical and sociological perspective. The realism is provided by the visual abilities that make it possible to grasp a certain element or structure as a separate object. This statement also corresponds to a certain minimal, internal ontological requirement for analyzing practice with the help of mathematical texts (Carter 2004, 2022, Chateaubriand 2012, Król 2003).

By attributing the set-theoretic modeling method to Dedekind (at least in that part of his mathematical practice), I do not deny his creative contribution, but I do point out that the theory he built required the development of a structure in such a way that the problems solved, the aspects selected and the methods of combining them remained in some connection with previously constructed (established) mathematical knowledge. In addition, I suggest a certain pragmatic ontology of the models introduced, as well as their parts, because in the models he built Dedekind referred to the visual aspects that allowed him to recognize particular structures as separate objects.

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