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REVIEW OF *IDENTITY AND INDISCERNIBILITY IN QUANTUM MECHANICS*

Tomasz Bigaj, *Identity and Indiscernibility in Quantum Mechanics*, Cham: Palgrave Macmillan 2022, pp. XV + 262, https://doi.org/10.1007/978-3-030-74870-8

Abstract

Tomasz Bigaj's new book is an authoritative and comprehensive discussion of recent issues concerning the means and metaphysical implications of individuating identical quantum particles. This review briefly summarises the book's contributions and considers some of its implications for the debate over indiscernibility in quantum mechanics.

Keywords: quantum mechanics, indiscernibility, Factorism

Tomasz Bigaj's *Identity and Indiscernibility in Quantum Mechanics* is an authoritative and comprehensive discussion of recent issues concerning the means of individuating identical quantum particles, and the metaphysical implications thereof. It will be of interest to philosophers of physics with interests in symmetry, representation, and the foundations of quantum mechanics; it could form a useful introduction to this literature for metaphysicians interested in the topic, albeit one with reasonably high technical prerequisites.

In broad terms, the book concerns three topics: how to "pick out" individual quantum particles in systems of many indistinguishable particles; the logic of

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discernibility; and the metaphysics of identity and individuality. For reasons of space, I will concentrate on what the book says about the first topic, since that is its primary focus.

To introduce the issues, recall that if two quantum systems are represented by Hilbert spaces H_1 and H_2, then the *joint system* comprised by the two systems together is represented by the *tensor product* Hilbert space $\mathrm{H}_1 \otimes \mathrm{H}_2$. Suppose now that the two systems are "indistinguishable particles." However precisely this is to be described, we take it to have as a consequence that $\mathrm{H}_1 = \mathrm{H}_2$, so that for any state $|\psi\rangle_1 \in \mathrm{H}_1$ there is a corresponding state $|\psi\rangle_2 \in \mathrm{H}_2$. This makes it possible to define a special kind of operator, namely a so-called permutation operator P_{12}. This acts on product states according to

(1) $$P_{12} : |\psi\rangle_1 \otimes |\phi\rangle_2 \mapsto |\phi\rangle_1 \otimes |\psi\rangle_2$$

whose action is then extended linearly to the whole of $\mathrm{H}_1 \otimes \mathrm{H}_2$. More generally, for a collection of N indistinguishable particles with Hilbert space $\mathrm{H}^N := \mathrm{H}_1 \otimes \cdots \otimes \mathrm{H}_N$, then for any permutation σ in the symmetric group S_N, the permutation operator P_σ acts on product states as

(2) $$P_\sigma : |\psi_1\rangle_1 \otimes \cdots \otimes |\psi_N\rangle_N \to |\psi_{\sigma(1)}\rangle_1 \otimes \cdots \otimes |\psi_{\sigma(N)}\rangle_N$$

and by linear extension on the whole of H^N.

A state $|\Psi\rangle \in \mathrm{H}^N$ is said to be *symmetric* if, for any $\sigma \in S_N$,

(3) $$P_\sigma |\Psi\rangle = |\Psi\rangle$$

and is said to be antisymmetric if, for any $\sigma \in S_N$,

(4) $$P_\sigma |\Psi\rangle = \mathrm{sgn}(\sigma)|\Psi\rangle$$

where $\mathrm{sgn}(\sigma)$ is the *sign* of the permutation: +1 or −1 according to whether the permutation is even or odd. The set of all symmetric states in H^N constitutes a subspace, as does the set of all antisymmetric states. When $N = 2$, these subspaces span the whole of H^N, but they do not do so for $N \geq 3$.

The above is all very standard, and (so far as it goes) philosophically uncontroversial. However, things start to get more interesting when we impose the *Symmetrisation Postulate* (SP), which Bigaj formulates as follows:

> For any system of [indistinguishable particles], its states are either exclusively symmetric, or exclusively antisymmetric. (2022: 24)

Thus, the SP insists that the only physically permissible states in H^N are those which lie in either the symmetric subspace or the antisymmetric subspace. Moreover, no process of dynamical evolution could move a state from one subspace to another (without passing through the forbidden region in-between). So, the SP has the corollary that if a collection of particles is described by a symmetric state at one time, it is described by a symmetric state at all times, and the same is true for antisymmetric states. This motivates the classification of particles as being either *bosons* or *fermions*, according to whether they form aggregates described by symmetric or antisymmetric states.

The central question of the book is how to interpret the joint states of indistinguishable particles, given the imposition of SP. Specifically, given such an (anti)symmetric joint state, what kinds of claims about the individual particles that comprise the system may be made? That is, what features of that formal mathematical object encode information about one particle rather than another?

The standard answer (which the book rejects) goes by the name of *Factorism*[1] and is stated by Bigaj as follows:

> In the N-fold tensor product of Hilbert spaces $H_1 \otimes \cdots \otimes H_N$ that is meant to represent states and properties of systems of N particles of the same type, and whose symmetric and antisymmetric sectors are assumed to contain all the admissible states of N bosons and N fermions respectively, each Hilbert space H_i represents states and properties of one individual particle. (2022: 32)

Of course, for *N distinguishable* systems, it is (reasonably) uncontroversial that the *i*th factor in the tensor product of those systems' Hilbert spaces represents the *i*th system. So, Factorism is the assertion that this interpretation should also be extended to the case of indistinguishable systems.

If Factorism is adopted, then one is led to the "orthodox" conclusion that indistinguishable quantum particles cannot be discerned from one another by their physical properties, and therefore quantum particles stand always and everywhere in violation of the Principle of the Identity of Indiscernibles (PII), as argued by Steven French and Michael Redhead (1988). However, if Factorism is rejected, then one can instead pursue the "heterodox" route of *physical* or *qualitative* individuation, whereby particles are identified by their physical properties and, consequently, may be discerned from one another. This heterodoxy, and the concomitant rejection of Factorism, have been discussed in several places in the literature,[2] but the greatest influences on Bigaj

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[1] This name was coined by Caulton (2014).

[2] See (Muller and Leegwater 2022, nn. 2-3) for a discussion of which papers have advo-

– by his acknowledgment – are Simon Saunders and Adam Caulton, especially (Caulton 2014).

It will be easiest to illustrate the orthodox and heterodox views with an example (one much discussed by Bigaj and in the literature). Suppose that we have a pair of electrons, and we factorise the Hilbert space H of each electron into Hilbert spaces representing spin and spatial degrees of freedom. Let $|\uparrow\rangle$ and $|\downarrow\rangle$ represent *z*-spin-up and *z*-spin-down states, and let $|L\rangle$ and $|R\rangle$ represent "localised in *L*" and "localised in *R*" states (where *L* and *R* do not overlap, so the states are orthogonal). Now consider the following state:

$$(5) \qquad \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|L\rangle_1 \otimes |\downarrow\rangle_2|R\rangle_2 - |\downarrow\rangle_1|R\rangle_1 \otimes |\uparrow\rangle_2|L\rangle_2\right).$$

Note that this state is anti-symmetric and therefore abides by the SP. Now consider the following question: can the two electrons described by the state (5) be discerned from one another?

No, says the orthodoxy. By Factorism, to discern them would be to identify some physical property that particle 1 exhibits but particle 2 lacks, but the very anti-symmetry of the state (5) precludes this: whatever can be said of particle 1 must apply equally to particle 2. As French and Redhead (1988: 240) put it, "*each* particle clearly 'partakes' of *both* the states $[|\uparrow\rangle|L\rangle]$ and $[|\downarrow\rangle|R\rangle]$ in the superposition of product states expressed in [(5)]."[3] And the same, of course, goes for any symmetric or antisymmetric state.

Yes, says the heterodoxy. No longer bound by Factorism, we instead individuate particles by their *physical descriptions*, rather than merely by the labels on Hilbert spaces. So, rather than "particle 1" and "particle 2," we can instead consider "the particle in *L*" and "the particle in *R*." So individuated, one can show (as the form of (5) suggests) that the particles can be discerned by their spin properties: the particle in *L* has spin-up, and that in *R* has spin-down. This indeed makes sense: if one were to perform a spin-measurement in *L*, then one would be guaranteed to get "up" as a result, and a spin-measurement in *R* is guaranteed to yield the result "down."

To see how this is done, in general, suppose we have a joint 2-particle state $|\Psi\rangle \in H \otimes H$; let A and S (respectively) be the anti-symmetric and symmetric

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cated for Factorism, and which have rejected it.

[3] Emphasis is in the original, though I have changed the notation: their state $|a^r\rangle$ corresponds to $|\uparrow\rangle|L\rangle$, and their $|a^s\rangle$ to $|\downarrow\rangle|R\rangle$. I don't believe this changes the sense of what they are saying, since their discussion is general and should include this example as a special case; but one should perhaps think of this as what someone inspired by French and Redhead might say about this case, not what they themselves would say.

subspaces of H ⊗ H and suppose further that E and F are orthogonal projectors on H. Then (says Bigaj, following Caulton) the particles are *physically discerned* from one another by E and F iff $|\Psi\rangle$ is an eigenvalue-1 eigenstate of the projector $K := E \otimes F + F \otimes E$.[4] Bigaj shows that if $|\Psi\rangle$ is a fermionic state, then such projectors E and F are bound to exist, but there exist bosonic states which cannot be discerned from one another in this way (for instance, product states of the form $|\phi\rangle_1 \otimes |\phi\rangle_2$).[5]

To talk about the properties of "the E-particle" and "the F-particle," Bigaj appeals to the following result from (Caulton 2014): the algebras of operators on K[A], on K[S], and on E[H] ⊗ F[H] are all unitarily equivalent. Thus, the collection of all (anti)symmetric states discerned by E and F may be regarded as equivalent to the collection of all states consisting of one E-particle and one F-particle. The fermionic representation of a proposition such as "the E-particle is B" (for some projector B) is then obtained by pulling back $B \otimes I$ from K[H] ⊗ F[H] to K[A]; pulling it back to K[S] gives the bosonic equivalent.

In the example above, the particles may be individuated by $I \otimes L$ and $I \otimes R$, where I is the identity operator on the spin Hilbert space, and L and R are projectors on the spatial Hilbert space, such that

(6) $L|L\rangle = |L\rangle$

(7) $L|R\rangle = 0 = R|L\rangle$

(8) $R|R\rangle = |R\rangle$

For brevity, let $H_L := (I \otimes L)[H]$ and $H_R := (I \otimes R)[H]$. It is then straightforward to show that the state (5) is an eigenstate of the projector obtained by pulling back $|\uparrow\rangle\langle\uparrow| \otimes I$ from H_L to $K[B]$. Thus, as promised, the state (5) satisfies the proposition "the L-particle is spin-up."

Overall, the case that Bigaj presents in favour of the heterodoxy is compelling. There do indeed seem to be good grounds for letting us make reference to particles by using their physical properties. Indeed, one might think that this is the *only* means by which reference to particles could be achieved. How else are

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[4] The heterodoxy interprets the projector K as the property that the joint system satisfies just in case it consists of one particle that satisfies the property E and one that satisfies the incompatible property F. The orthodoxy does not, since the E-ness and F-ness are not appropriately associated with the factor spaces in H ⊗ H.

[5] Thus, bosons still present a counterexample to the PII, at least when they are in such states.

we to refer to one particle rather than another if not by description? Ostension will hardly do as a general solution, given the macroscopic dimensions of anything with which we might do the pointing; in any event, it seems to presuppose that the particles will be individuated by location. Baptising one particle rather than another presumably also requires interacting with it, which would therefore require that its physical properties are distinctive enough that we can determinately baptise *it* and not one of its kin.

However, this same thought does lead to a bit of a worry for a rejection of Factorism that is as full-throated as Bigaj's. Certainly, one might think, one shouldn't treat Hilbert-space labels associated with particles as significant *in general*. However, does this demonstrate that one can *never* associate particular Hilbert-space factors with individual particles? On the contrary, it seems clear that one can: as discussed above, when a pair of particles may be individuated by projectors E and F, their state can be represented on the Hilbert space $E[\mathrm{H}] \otimes F[\mathrm{H}]$. It is then fairly natural to think of these two factors as being, respectively, the Hilbert space of the E-particle and the Hilbert space of the F-particle. The joint electron state is then represented by a (nonsymmetric!) state in a tensor-product Hilbert space whose factors *do* correspond to the different particles – precisely because the factors are each tied to individual discerning properties.

Indeed, in the introductory chapter Bigaj seems to suggest this application of the heterodoxy. After quoting a remark of Claude Cohen-Tannoudji, Bernard Diu, Franck Laloë (1978) that "under certain special conditions, identical particles behave as if they were actually different, and it is not necessary to take the symmetrization postulate into account in order to obtain correct physical predictions" (Cohen-Tannoudji, Diu, Laloë 1978: 1406, quoted: Bigaj 2022: 3-4), Bigaj continues

> The quoted fragment is baffling. How can indiscernible objects "behave" as if they were discernible, even under "certain special conditions"? . . . It is difficult to make sense of a situation in which entirely indistinguishable objects behave as if they were distinguishable, unless we make some crucial changes in the way we identify these objects. And it turns out that this may be the key to understanding the above-mentioned quote: perhaps what justifies the suspension of the symmetrization postulate is an alternative method of "carving up" the totality of the composite system into smaller components, so that these new components not only behave "as if" they were distinguishable, but really are. (2022: 4)

This proposal is, I take it, what F. A. Muller and Gijs Leegwater (2022) refer to as *Descriptive Factorism*. So, the question arises: has Bigaj come to bury Factorism, or to praise it? In response, Bigaj argues that

> Factorism presupposed by orthodoxy (and required for the Indiscernibility Thesis [of French and Redhead]) contains the additional assumption that the factor Hilbert spaces corresponding to individual particles must figure in the original Symmetrization Postulate restricting the available states to symmetric/antisymmetric sectors of the whole product. Thus the rewriting of the states of same-type particles in the tensor product of individuating blocks [E[H] ⊗ F[H]] does not reinstate this interpretation of Factorism. (2022: 122, n. 5.12)

This is certainly true. In particular, Descriptive Factorism will not suffice to rehabilitate French and Redhead's argument that the PII is always violated by quantum particles: if we can refer to two particles as "the *E*-particle" and "the *F*-particle," then we can also predicate different properties of them (for instance, *E* and *F*).

However, it would have been interesting to see a more thorough exploration of this revised version of Factorism. In particular, it strikes me that Descriptive Factorism offers an explanation of why Hilbert-space labels are permissible in the case of distinguishable particles; indeed, Descriptive Factorism goes some way towards breaking down the distinction between distinguishable and indistinguishable particles. For example, suppose that one decided to represent both a proton and an electron in a *single* Hilbert space: one which is a tensor product between a two-dimensional Hilbert space H^{σ} representing spin, and a two-dimensional Hilbert space H^{κ} representing charge and mass (for simplicity, ignoring spatial degrees of freedom). Thus, we suppose that there are orthogonal simultaneous eigenstates $|e\rangle$ and $|p\rangle$ of compatible mass and charge operators on H^{κ}, where the eigenvalues of $|e\rangle$ are the electron mass and charge, and those of $|p\rangle$ are the proton mass and charge. Finally, we assume that the mass and charge operators commute with the Hamiltonian, i.e., that electrons cannot turn into protons, and vice versa.

Given this setup, the electron and proton are just as "indistinguishable" from one another as two electrons would be: there is nothing intrinsically different between them, only differences in state. This suggests that the SP should be applied so that a joint electron-proton state could be (say)

$$\text{(9)} \qquad \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|e\rangle_1 \otimes |\downarrow\rangle_2|p\rangle_2 - |\downarrow\rangle_1|p\rangle_1 \otimes |\uparrow\rangle_2|e\rangle_2\right).$$

This state allows individuation by the properties "having the mass and charge of an electron" (i.e., $I \otimes |e\rangle\langle e|$) and "having the mass and charge of a proton" (i.e., $I \otimes |p\rangle\langle p|$). As with the state (5), upon this individuation we find that the spin properties of each thus-individuated particle are determinate: the electron is spin-up and the proton is spin-down. Since these individuating properties are dynamically preserved, they may be used throughout the particles' lifetimes to secure reference. Hence, we lose nothing in passing to a tensor-product Hilbert space $H_e \otimes H_p$, where $H_e = H^\sigma \otimes (|e\rangle\langle e|[H^\kappa]) \cong H^\sigma$, and the same (*mutatis mutandis*) for H_p.

Thus, we have legitimated the practice of writing the joint electron-proton system's state as an element of a tensor-product Hilbert space, where each factor of the Hilbert space is spin Hilbert space – in other words, precisely what we would have done otherwise. Conversely, and as Bigaj notes in his *Introduction*, if we have two indistinguishable particles that are discernible throughout the period we are interested in, then – by the same logic – we can represent their joint state as a (nonsymmetric) state in a tensor-product Hilbert space whose factors correspond to the discerning properties. So, the very distinction between distinguishable and indistinguishable particles starts to look rather more blurry: it is more a matter of perspective than a hard and fast metaphysical difference. That said, I emphasize again that this is likely an argument with which Bigaj would agree; to a large extent, it reflects the fact that I was sufficiently convinced by the case against the unvarnished version of Factorism[6] that I now want to know about the subtleties of more refined versions.

There is much more in the book than this review has covered: in particular, as mentioned at the start, a great deal of material on the logic of discernibility, as well as a thorough discussion of metaphysical topics related to these issues. Overall, this book is likely to become a standard reference for those wishing to get a handle on questions of identity and indiscernibility in the quantum realm.

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[6] What Muller and Leegwater (2022) call Direct Factorism.

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